

\mathbb{R} -orbit Reflexive Operators

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December 9, 2011

Abstract

We completely characterize orbit reflexivity and \mathbb{R} -orbit reflexivity for matrices in $\mathcal{M}_N(\mathbb{R})$. Unlike the complex case in which every matrix is orbit reflexive and \mathbb{C} -orbit reflexivity is characterized solely in terms of the Jordan form, the orbit reflexivity and \mathbb{R} -orbit reflexivity of a matrix in $\mathcal{M}_N(\mathbb{R})$ is described in terms of the linear dependence over \mathbb{Q} of certain elements of \mathbb{R}/\mathbb{Q} . We also show that every $n \times n$ matrix over an uncountable field \mathbb{F} is algebraically \mathbb{F} -orbit reflexive.

1 Introduction

The term *reflexive operator* was coined by P. R. Halmos [20], and studied by many authors, e.g., [1], [2], [3], [4], [5], [6], [7], [9], [12], [13], [16], [17], [18], [21], [22], [24], [25], [26], [30], [31], [33]. It was proved by J. Deddens and P. Fillmore [7] that an $n \times n$ complex matrix T is reflexive if and only if, for each eigenvalue λ of T , the two largest Jordan blocks corresponding to λ in the Jordan canonical form of T differ in size by at most 1. Later, D. Hadwin [12] characterized (algebraic) reflexivity for an $n \times n$ matrix over an arbitrary field; in this setting the analog of the Jordan form contains blocks, which we will still call Jordan blocks, of the form

$$J_m(A) = \begin{pmatrix} A & I & 0 & \cdots & 0 \\ 0 & A & I & \ddots & \vdots \\ 0 & 0 & A & \ddots & 0 \\ \vdots & \vdots & \ddots & A & I \\ 0 & 0 & \cdots & 0 & A \end{pmatrix},$$

where A is the companion matrix of an irreducible factor of the minimal polynomial for T . When the irreducible factor has degree 1, the matrix A is 1×1 and an eigenvalue of T . Hadwin [12] proved that an $n \times n$ matrix T over a field

\mathbb{F} is (algebraically) reflexive if, for each eigenvalue of T , the two largest Jordan blocks differ in size by at most 1, and for an irreducible factor of the minimal polynomial of T that has degree greater than 1, the two largest Jordan blocks have the same size.

D. Hadwin, E. A. Nordgren, H. Radjavi and P. Rosenthal [19] introduced the notion of an *orbit-reflexive operator*. They proved that on a Hilbert space this class includes all normal operators, algebraic operators, compact operators, contractions and unilateral weighted shift operators. It was over twenty years before examples were constructed [10] and [29] (see also [8]) of operators that are not orbit reflexive. V. Müller and J. Vršovský [29] proved that if $r(T) \neq 1$ ($r(T)$ denotes the spectral radius of T), then T is orbit reflexive. In [14], where the notion of *null-orbit reflexive operator* was introduced, the authors proved that every polynomially bounded operator on a Hilbert space is orbit reflexive.

Recently, M. McHugh and the authors [15], [27] introduced the notion of \mathbb{C} -orbit reflexivity and \mathbb{R} -orbit reflexivity, and they proved that an $n \times n$ complex matrix T is \mathbb{C} -orbit reflexive if and only if it is nilpotent or, among all the Jordan blocks corresponding to all eigenvalues with modulus equal to the spectral radius $r(T)$ of T , the two largest blocks differ in size by at most 1.

In this paper we address orbit reflexivity and \mathbb{R} -orbit reflexivity for a matrix in $\mathcal{M}_n(\mathbb{R})$. In $\mathcal{M}_n(\mathbb{C})$ every matrix is orbit reflexive and \mathbb{C} -orbit reflexivity is characterized solely in terms of the Jordan form. Surprisingly, neither of these facts remain true for $\mathcal{M}_n(\mathbb{R})$; the characterizations involve a little number theory, i.e., linear dependence over \mathbb{Q} of elements in \mathbb{R}/\mathbb{Q} .

2 Algebraic Results

An irreducible factor $p(x)$ of a polynomial in $\mathbb{R}[x]$ has degree at most 2. If $p(x) \in \mathbb{R}[x]$ is monic and irreducible and $\deg p = 2$, then p has roots $\alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, $p(x) = (x - \alpha)^2 + \beta^2$, and the corresponding companion matrix looks like $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, where

$$\alpha + i\beta = re^{i\theta}$$

with $r = \sqrt{\alpha^2 + \beta^2}$ and $0 \leq \theta < 2\pi$. The matrix

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

acts on \mathbb{R}^2 as a counterclockwise rotation by the angle θ . More generally, if we identify \mathbb{R}^2 with \mathbb{C} , then $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ acts as multiplication by $\alpha + i\beta$. An $m \times m$ Jordan block corresponding to $A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, is given by $J_m(A)$. However, $J_m(A)$ is similar to $rJ_m(R_\theta)$, and we will represent the Jordan blocks this

way. A Jordan block J of T *splits*, or, *is splitting*, if the irreducible polynomial associated to it has degree 1, i.e., it corresponds to a real eigenvalue of T .

Since a real matrix may have empty spectrum, we let $\sigma_p(T)$ denote the *point spectrum* of T , the set of real eigenvalues of T . Note that $\sigma_p(T) = \emptyset$ is possible. We define the *spectral radius* to be

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}},$$

which is the spectral radius of T considered as a matrix in $\mathcal{M}_n(\mathbb{C})$. Note that $r(J_m(R_\theta)) = 1$ and $r\left(\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\right) = \sqrt{\alpha^2 + \beta^2}$.

If X is a vector space over a field \mathbb{F} , and T is a linear transformation on X , then $\mathcal{P}_{\mathbb{F}}(T) = \{p(T) : p \in \mathbb{F}[t]\}$. A *linear manifold* M in X , is the translate of a linear subspace, i.e., nonempty subset M so that when $x \in M$, $M - x$ is a linear subspace.

We begin with a lemma on the cardinality of the field. In the case where the field is \mathbb{R} or \mathbb{C} , the lemma is an immediate consequence of the Baire category theorem.

Lemma 1 *If \mathbb{F} is an uncountable field and n is a positive integer, then \mathbb{F}^n is not a countable union of proper linear subspaces.*

Proof. Let $S = \{(1, x, x^2, \dots, x^{n-1}) : x \in \mathbb{F}\}$. Since any n distinct elements of S are linearly independent, the intersection of any proper linear subspace with S has cardinality at most $n - 1$. However, S is uncountable, so S is not contained in a countable union of proper linear subspaces of \mathbb{F}^n . ■

Theorem 2 *If \mathbb{F} is an uncountable field, then every $T \in \mathcal{M}_N(\mathbb{F})$ is algebraically \mathbb{F} -orbit reflexive and algebraically orbit-reflexive.*

Proof. It is known from [16] that $\text{AlgLat}_0(T) \cap \{T\}' = \mathcal{P}_{\mathbb{F}}(T)$, and that this algebra of operators has a separating vector e . We know from [15] that every nilpotent matrix is algebraically \mathbb{F} -orbit reflexive. Suppose A is an invertible $k \times k$ matrix and $S \in \mathbb{F}\text{-OrbRef}_0(A)$. Then, for every $x \in \mathbb{F}^k$, there is a $\lambda \in \mathbb{F}$ and an $m \geq 0$ such that $Sx = \lambda A^m x$. Hence,

$$\mathbb{F}^k = \bigcup_{m=0}^{\infty} \bigcup_{\lambda \in \sigma_p(A^{-m}S)} \text{Ker}(A^{-m}S - \lambda),$$

which, by Lemma 1, implies there is an $m \geq 0$ and a $\lambda \in \mathbb{F}$ such that $S = \lambda A^m$. Hence A is algebraically \mathbb{F} -orbit reflexive. Since every $T \in \mathcal{M}_n(\mathbb{F})$ is the direct sum of a nilpotent matrix N and an invertible matrix A , it follows that every $S \in \mathbb{F}\text{-OrbRef}_0(T)$ is a direct sum of αN^s and βA^t for $\alpha, \beta \in \mathbb{F}$ and integers $s, t \geq 0$. It follows that $S \in \text{AlgLat}_0(T) \cap \{T\}'$; whence there is a polynomial

$p \in \mathbb{F}[x]$ such that $S = p(T)$. However, there is a $\lambda \in \mathbb{F}$ and an $m \geq 0$ such that

$$p(T)e = Se = \lambda T^m e.$$

Since e is separating for $\mathcal{P}(T)$, we see that $S = p(T) = \lambda T^m$, which implies T is \mathbb{F} -orbit reflexive. The proof that T is algebraically orbit reflexive is very similar. ■

Corollary 3 *If $T \in \mathcal{M}_n(\mathbb{R})$ and $\{T^k : k \geq 0\}$ is finite, e.g., $T^N = I$ or $T^N = 0$ for some positive integer N , then $\mathbb{R}\text{-OrbRef}(T) = \mathbb{R}\text{-OrbRef}_0(T) = \mathbb{R}\text{-Orb}(T)$ and $\text{OrbRef}(T) = \text{OrbRef}_0(T) = \text{Orb}(T)$.*

Proof. Since $\{T^k : k \geq 0\}$ is finite, we know, for every vector x , that $\mathbb{R}\text{-Orb}(T)x$ and $\text{Orb}(T)x$ are closed, implying $\mathbb{R}\text{-OrbRef}(T) = \mathbb{R}\text{-OrbRef}_0(T)$ and $\text{OrbRef}(T) = \text{OrbRef}_0(T)$. ■

Corollary 4 *If $T \in \mathcal{M}_n(\mathbb{R})$, $T = A \oplus B$ with $A^N = I$ for some minimal $N \geq 1$ and $r(B) < 1$, then T is \mathbb{R} -orbit reflexive.*

Proof. Suppose $S \in \mathbb{R}\text{-OrbRef}(T)$. Then $S = S_1 \oplus S_2$ and, by Corollary 3, we know that $S_1 = \lambda A^s$ for some $\lambda \in \mathbb{R}$ and some $s \geq 0$. If $S_1 = 0$ it easily follows by considering $x \oplus y$ with $x \neq 0$ and y arbitrary, that $S_2 = 0$, which implies $S = 0$. Hence we can assume that $S_1 \neq 0$.

Note that

$$S_1^N = \lambda^N (A^N)^s = \lambda^N.$$

Let $E = \{e^{2\pi i k/n} \lambda : k = 1, \dots, n\}$. Choose a separating unit vector x_0 for $\mathcal{P}_R(A)$. If $S_1 x_0 = \lambda_1 A^t x_0$, we have $S_1 = \lambda_1 A^t$, which implies $\lambda_1 \in E$. Suppose y is in the domain of B , then there is a sequence $\{k_m\}$ of positive integers and a sequence $\{\beta_m\}$ in \mathbb{R} such that

$$\beta_m T^{k_m} (x_0 \oplus y) \rightarrow S_1 x_0 \oplus S_2 y.$$

We have $\beta_m A^{k_m} x_0 \rightarrow \lambda A^s x_0$, which implies $\{\beta_m\}$ is bounded. If $\{k_m\}$ is unbounded, then it has a subsequence diverging to ∞ , which implies $S_2 y = 0$, since $\|B^k\| \rightarrow 0$ as $k \rightarrow \infty$. If $\{k_m\}$ is bounded, then it has a subsequence $\{k_{m_j}\}$ with a constant value t , and we get $\beta_{m_j} \rightarrow \lambda_1$ for some $\lambda_1 \in E$. Hence the domain of B is a countable union,

$$\ker S_2 \cup \bigcup_{k \in \mathbb{N}, \gamma \in E} \ker (S_2 - \gamma B^k).$$

It follows from Lemma 1 that $S_2 \in \mathcal{P}_{\mathbb{R}}(B)$. If we choose a vector y_0 that is separating for $\mathcal{P}_{\mathbb{R}}(B)$, we see from $S(x_0 \oplus y_0) \in [\mathbb{R}\text{-Orb}(T)(x_0 \oplus y_0)]^-$, that $S \in \mathbb{R}\text{-Orb}(T)$. ■

3 Main Results

A key ingredient in our proofs is the following well-known result from number theory. We sketch the elementary proof for completeness. For notation we let $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$ be the unit circle, \mathbb{T}^k a direct product of k copies of \mathbb{T} , and $\mu_k = \mu \times \cdots \times \mu$ be Haar measure on \mathbb{T}^k , where μ is normalized arc length on \mathbb{T} . If $\lambda = (z_1, \dots, z_k) \in \mathbb{T}^k$ and we define

$$\lambda^n = (z_1^n, \dots, z_k^n)$$

for $n = 0, 1, 2, \dots$.

Lemma 5 *Suppose $\theta_1, \dots, \theta_k \in \mathbb{R}$, and let $\lambda = (e^{i\theta_1}, \dots, e^{i\theta_k})$. The following are equivalent:*

1. $\{\lambda, \lambda^2, \dots\}$ is dense in \mathbb{T}^k ,
2. $\{1, \theta_1/2\pi, \dots, \theta_k/2\pi\}$ is linearly independent over \mathbb{Q} ,
3. for every $f \in C(\mathbb{T}^k)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\lambda^n) = \int_{\mathbb{T}^k} f d\mu_k.$$

Proof. If $f(z_1, \dots, z_k) = z_1^{m_1} \cdots z_k^{m_k}$ for integers m_1, \dots, m_k , then statement (2) is equivalent to saying $f(\lambda) \neq 1$ whenever $(m_1, \dots, m_k) \neq (0, \dots, 0)$. For such a monomial f we know that $\int_{\mathbb{T}^k} f d\mu_k = 0$, and we know that $f(\lambda^n) = f(\lambda)^n$ for $n \geq 1$. Thus statement (2) implies that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\lambda^n) = \lim_{N \rightarrow \infty} \frac{1}{N} \frac{1 - f(\lambda)^N}{1 - f(\lambda)} f(\lambda) \rightarrow 0 = \int_{\mathbb{T}^k} f d\mu_k.$$

It follows from the Stone-Weierstrass theorem that the span of the monomials is dense in $C(\mathbb{T}^k)$, so we see that (2) \implies (3). On the other hand (3) implies that, for every nonnegative continuous function f vanishing on $\{\lambda, \lambda^2, \dots\}$ we must have $\int_{\mathbb{T}^k} f d\mu_k = 0$, which implies $f = 0$. If $x \in \mathbb{T}^k \setminus \{\lambda, \lambda^2, \dots\}^-$, there is a nonnegative continuous function f vanishing on $\{\lambda, \lambda^2, \dots\}$ with $f(x) \neq 0$. Hence (3) \implies (1). If f is a nonconstant monomial and $f(\lambda) = 1$, then the closure of $\{\lambda, \lambda^2, \dots\}$ is contained in $f^{-1}(\{1\})$, which proves that (1) \implies (2).

■

The next two results show that in $\mathcal{M}_N(\mathbb{R})$ orbit reflexivity is not the same as in $\mathcal{M}_N(\mathbb{C})$.

Lemma 6 *Suppose $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in [0, 2\pi)$, and $T \in \mathcal{M}_N(\mathbb{R})$ is a direct sum of $R_{\theta_1} \oplus \cdots \oplus R_{\theta_k} \oplus B \oplus C$ with $B^2 = 1$ and $r(C) < 1$. (The summands B and C might not be present.) The following are equivalent:*

1. T is orbit reflexive
2. T is \mathbb{R} -orbit reflexive
3. There are nonzero integers s_1, \dots, s_k and an integer t such that

$$\sum_{j=1}^k s_j \theta_j = 2\pi t.$$

4. For every $j \in \{1, \dots, k\}$, $\theta_j/2\pi \in sp_{\mathbb{Q}}(\{1\} \cup \{\theta_i/2\pi : 1 \leq i \neq j \leq k\})$.

Proof. The equivalence of (4) and (3) is easy.

(1) \implies (4) and (2) \implies (4). Assume (4) is false. We can assume that

$$\theta_1/2\pi \notin sp_{\mathbb{Q}}(\{1\} \cup \{\theta_i/2\pi : 2 \leq i \leq k\}).$$

We can assume that $\{1, \theta_2/2\pi, \dots, \theta_s/2\pi\}$ is a basis for the linear span over \mathbb{Q} of $\{1\} \cup \{\theta_i/2\pi : 2 \leq i \leq k\}$, which makes $\theta_1/2\pi, \theta_2/2\pi, \dots, \theta_s/2\pi$ irrational, and makes $\{1, \theta_1/2\pi, \dots, \theta_s/2\pi\}$ linearly independent over \mathbb{Q} . Since each $\theta_j/2\pi$, $s < j \leq k$ is a rational linear combination of $1, \theta_2/2\pi, \dots, \theta_s/2\pi$, there is a positive integer d such that, for $s < j \leq k$, each $d\theta_j/2\pi$ is an integral linear combination of $1, \theta_2/2\pi, \dots, \theta_s/2\pi$. Suppose $\alpha \in [0, 2\pi)$. Since $\{1, \theta_1/4\pi d, \dots, \theta_s/4\pi d\}$ is linearly independent over \mathbb{Q} , it follows from Lemma 5 that there is a sequence $\{m_n\}$ of positive integers such that $m_n \rightarrow \infty$,

$$R_{\theta_1}^{m_n} = R_{m_n \theta_1} \rightarrow R_{\alpha/2d},$$

$$R_{\theta_j}^{m_n} = R_{m_n \theta_j} \rightarrow I$$

for $2 \leq j \leq s$. This implies that $R_{\theta_1}^{2dm_n} = R_{2dm_n \theta_1} \rightarrow R_{\alpha}$ and $R_{\theta_j}^{2dm_n} = R_{2dm_n \theta_j} \rightarrow I$ for $2 \leq j \leq s$. If $s < j \leq k$, there are integers t_2, \dots, t_s and t such

that $d\theta_j = t2\pi + \sum_{i=2}^s t_i \theta_i$, which implies

$$R_{\theta_j}^{2dm_n} = I^{2tm_n} \prod_{i=2}^s (R_{m_n \theta_i})^{2t_i} \rightarrow I.$$

Moreover,

$$(B \oplus C)^{2dm_n} = B^{2dm_n} \oplus C^{2dm_n} \rightarrow I \oplus 0 = P.$$

Let $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and define $S = F \oplus I \oplus \dots \oplus I \oplus P$. It follows from the fact that, for every $x \in \mathbb{R}^2$ there is an $\alpha \in [0, 2\pi)$ such that $Fx = R_{\alpha}x$, that $S \in \text{OrbRef} \subseteq \mathbb{R}\text{-OrbRef}(T)$. Since $FR_{\theta_1} \neq R_{\theta_1}F$ (because $\sin \theta_1 \neq 0$), it follows that $ST \neq TS$, and we see that both (1) and (2) are false.

(3) \implies (2). Suppose (3) is true. If $k = 1$, then $\theta_1/2\pi \in \mathbb{Q}$, and $R_{\theta_1}^N = I$ for some positive integer N , which, by Corollary 4, implies T is \mathbb{R} -orbit reflexive.

Hence we can assume $k \geq 2$, which, by (3), implies $\theta_1/2\pi \notin \mathbb{Q}$. Suppose $S \in \mathbb{R}\text{-OrbRef}(T)$. Since $\mathbb{R}\text{-OrbRef}(T)$ is contained in $\text{AlgLat}(T)$, we can write $S = S_1 \oplus \cdots \oplus S_k \oplus D \oplus E$. Suppose $x \neq 0$ is in the domain of S_1 . We consider two cases:

Case 1. $S_1x = 0$. If y is any vector orthogonal to the domain of S_1 , there is a sequence $\{m_n\}$ of nonnegative integers and a sequence $\{\lambda_n\}$ in \mathbb{R} such that $S(x \oplus y) = \lim \lambda_n T^{m_n}(x \oplus y)$. Thus $|\lambda_n| \|x\| \rightarrow \|S_1x\| = 0$, which implies $\lambda_n \rightarrow 0$, and since $\{\|T^n\|\}$ is bounded, we see that $S(x \oplus y) = 0$. Thus $0 = S_2 = \cdots = S_k$ and $D = 0, E = 0$. Since $k \geq 2$, and arguing as above (when we showed $S_1 = 0 \implies S_2 = 0$), we know $S_1 = 0$, and thus $S = 0$.

Case 2. $S_1x \neq 0$. Let $x_1 = x$, and choose x_j in the domain of S_j for $2 \leq j \leq k$ with each $\|x_j\| = \|x\|$, and let $u = x \oplus x_2 \oplus \cdots \oplus x_k \oplus 0 \oplus 0$. Since $R_{\theta_1} \oplus \cdots \oplus R_{\theta_k}$ is an isometry and $S \in \mathbb{R}\text{-OrbRef}(T)$, it follows that there is a sequence $\{m_n\}$ of nonnegative integers and a sequence $\{\lambda_n\}$ in \mathbb{R} such that $0 \neq Su = \lim_{n \rightarrow \infty} \lambda_n T^{m_n} u$. Hence, $\{\lambda_n\}$ is bounded, so we can assume that $\lambda_n \rightarrow \lambda$ for some nonzero $\lambda \in \mathbb{R}$, and we can assume that $T^{m_n} \rightarrow R_{\alpha_1} \oplus \cdots \oplus R_{\alpha_k} \oplus F \oplus G$ with $0 \leq \alpha_1, \dots, \alpha_k < 2\pi$. We know that $|\lambda| = \|S_1x\| \neq 0$, and, for $1 \leq j \leq k$, $S_jx_j = \|S_1x\| R_{\alpha_j}x_j$ if $\lambda > 0$ and $S_jx_j = \|S_1x\| R_{\alpha_j+\pi}x_j$ if $\lambda < 0$. Moreover,

since $R_{\theta_j}^{m_n} \rightarrow R_{\alpha_j}$ for $1 \leq j \leq k$, we have, from (3), that $\sum_{j=1}^k s_j \alpha_j \in 2\pi\mathbb{Z}$, and

thus $\sum_{j=1}^k s_j (\alpha_j + \pi) \in \pi\mathbb{Z}$. Suppose now we replace x_1 with another vector y in

the domain of S_1 with $\|y\| = \|x_1\|$, we get real numbers β_1, \dots, β_k such that $S_1y = \|S_1y\| R_{\beta_1}y$ and $S_jx_j = \|S_1y\| R_{\beta_j}x_j = \|S_1y\| R_{\alpha_j}x_j$ for $2 \leq j \leq k$, and

such that $\sum_{j=1}^k s_j \beta_j \in \pi\mathbb{Z}$. However, for $2 \leq j \leq k$, we must have $\beta_j - \alpha_j \in \pi\mathbb{Z}$.

Hence, $s_1\beta_1 - s_1\alpha_1 \in \pi\mathbb{Z}$. Hence the domain of S_1 is the union

$$\bigcup_{n \in \mathbb{Z}} \ker(S_1 - \|S_1x\| R_{\alpha_1+n\pi/s_1}),$$

which, by Lemma 1, implies that there is a $\gamma_1 \in [0, 2\pi) \cap \left(\alpha_1 + \frac{\pi}{s_1}\mathbb{Z} + 2\pi\mathbb{Z}\right)$ such that $S_1 = \|S_1x\| R_{\gamma_1}$. Similarly, we get, for $2 \leq j \leq k$, that $S_j = \|S_1x\| R_{\gamma_j}$ for some $\gamma_j \in [0, 2\pi)$.

Applying the same reasoning we see that $D = \|S_1x\| B$ or $D = -\|S_1x\| B$. Also, for every f in the domain of C we get $Ef \in \mathbb{R}\text{-Orb}(C)f$, so, by Theorem 2, $E \in \mathbb{R}\text{-Orb}(C)$. We therefore have $S_j \in \mathcal{P}_{\mathbb{R}}(R_{\theta_j})$ for $1 \leq j \leq k$, $D \in \mathcal{P}_{\mathbb{R}}(B)$, and $E \in \mathcal{P}_{\mathbb{R}}(C)$. If we choose separating vectors v_j for each $\mathcal{P}_{\mathbb{R}}(R_{\theta_j})$ ($1 \leq j \leq k$) and w_1 for $\mathcal{P}_{\mathbb{R}}(B)$ and w_2 for $\mathcal{P}_{\mathbb{R}}(C)$, and we let $\eta = v_1 \oplus \cdots \oplus v_k \oplus w_1 \oplus w_2$, then there is a sequence $\{q_n\}$ of nonnegative integers and a sequence $\{t_n\}$ in \mathbb{R} such that

$$t_n T^{q_n} \eta \rightarrow S\eta,$$

and it follows that

$$t_n T^{q_n} \rightarrow S.$$

Thus $S \in \mathbb{R}\text{-Orb}(T)^{-SOT}$.

(2) \implies (1). Suppose (2) is true, let e be a separating vector for $\mathcal{P}_{\mathbb{R}}(T)$, and suppose $S \in \text{OrbRef}(T) \subseteq \mathbb{R}\text{-OrbRef}(T) = \mathbb{R}\text{-Orb}(T) \subseteq \mathcal{P}_{\mathbb{R}}(T)$ (by (2)). Since there is a sequence $\{m_n\}$ of nonnegative integers such that $T^{m_n}e \rightarrow Se$, it follows that $T^{m_n} \rightarrow S$. Hence (1) is proved. ■

Theorem 7 *A matrix $T \in \mathcal{M}_N(\mathbb{R})$ fails to be orbit reflexive if and only if it is similar to a matrix of the form in Lemma 6 that is not orbit reflexive.*

Proof. We know from [14, Lemma 17] that if one of the sets $\{x \in \mathbb{R}^N : T^k x \rightarrow 0\}$ or $\{x \in \mathbb{R}^N : \|T^k x\| \rightarrow \infty\}$ is not a countable union of nowhere dense subsets of \mathbb{R}^N , then T is orbit reflexive. Thus if $r(T) < 1$, then T is orbit reflexive. If $r(T) > 1$, then the Jordan form shows that $\{x \in \mathbb{R}^N : \|T^k x\| \rightarrow \infty\}$ has nonempty interior, which implies T is orbit reflexive. Hence we are left with the case where $r(T) = 1$. Moreover, if the Jordan form of T has an $m \times m$ block of

the form $\begin{pmatrix} A & I_2 & \cdots & 0 \\ 0 & A & \ddots & \vdots \\ \vdots & 0 & \ddots & I_2 \\ 0 & \cdots & 0 & A \end{pmatrix}$ with $A = \pm I$ or $A = R_\theta$, then for any vector

$x \in \mathbb{R}^N$ whose m^{th} -coordinate relative to this summand is nonzero, we have $\|T^k x\| \rightarrow \infty$; whence T is orbit reflexive. Thus the Jordan form of a matrix that is not orbit reflexive must be as the matrix in Lemma 6. ■

If X is a Banach space over \mathbb{R} , and $T \in B(\mathbb{R})$ is algebraic, i.e., there is a nonzero polynomial $p \in \mathbb{R}[x]$ such that $p(T) = 0$, then, as a linear transformation, T has a Jordan form with finitely many distinct blocks, but possibly with some of the blocks having infinite multiplicity.

Corollary 8 *Suppose X is a Banach space over \mathbb{R} and $T \in B(X)$ is algebraic. Then T fails to be orbit-reflexive if and only if $r(T) = 1$, and the Jordan form for T has one block R_{θ_1} of multiplicity 1, other blocks of the form $R_{\theta_2}, \dots, R_{\theta_k}$ with $\theta_1/2\pi \notin \text{sp}_{\mathbb{Q}}\{1, \theta_2, \dots, \theta_k\}$, the remaining blocks of the form $\pm I$ or blocks with spectral radius less than 1.*

Proof. Suppose T has the indicated form. Then there is an invertible operator $D \in B(X)$ such that $D^{-1}TD = R_{\theta_1} \oplus A \oplus B$ with $r(A) = 1$ and $r(B) < 1$. Let $S = F \oplus 1 \oplus 0$. Suppose $x \in X$. Choose a finite-dimensional invariant subspace M for T of the form $M = M_1 \oplus M_2 \oplus M_3$, with M_1 equal to the domain of S_1 such that $x \in M$. It follows from the assumptions on T and the proof of Theorem 7 that $S|M \in \text{OrbRef}(T|M)$. In particular, Sx is in the closure $\text{Orb}(T)x$. Thus $S \in \text{OrbRef}(T)$, but $ST \neq TS$, so T is not orbit reflexive.

On the other hand, if T does not have the described form, then, given $S \in \text{OrbRef}(T)$, vectors x_1, \dots, x_n and $\varepsilon > 0$, there is a finite-dimensional invariant subspace E of X containing x_1, \dots, x_n such that $T|_E$ is orbit reflexive because of the conditions in Theorem 7. Hence, since $S|_E \in \text{OrbRef}(T|_E)$, there is an integer $m \geq 0$ such that

$$\|Sx_j - T^m x_j\| < \varepsilon$$

for $1 \leq j \leq n$. Thus S is in the strong operator closure of $\text{Orb}(T)$. Thus T is orbit reflexive. ■

Theorem 9 *A matrix $T \in \mathcal{M}_N(\mathbb{R})$ fails to be \mathbb{R} -orbit reflexive if and only if $r(T) \neq 0$ with the largest size of a Jordan block with spectral radius $r(T)$ being m , and either*

1. *every Jordan block of T with spectral radius $r(T)$ splits over \mathbb{R} , and the largest two such blocks differ in size by more than 1, or*
2. *there exist $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in [0, 2\pi)$ such that the direct sum of the non-splitting $m \times m$ Jordan blocks of $T/r(T)$ that have spectral radius 1 is similar to*

$$J_m(R_{\theta_1}) \oplus \dots \oplus J_m(R_{\theta_k})$$

with $\theta_1/2\pi \notin \text{sp}_{\mathbb{Q}}\{1, \theta_2/2\pi, \dots, \theta_k/2\pi\}$.

Proof. We know that if $r(T) = 0$, then T is nilpotent, which, by Corollary 3, implies T is \mathbb{R} -orbit reflexive. Hence we can assume that $r(T) > 0$. Replacing T by $T/r(T)$, we can, and do, assume $r(T) = 1$.

In the case where every Jordan block of T with spectral radius $r(T)$ splits, the proof that T is not \mathbb{R} -orbit reflexive is equivalent to the condition in (1) is exactly the same at the proof of Theorem 7 in [15].

Next suppose T satisfies (2). Then, as in the proof of (1) \implies (4) in Lemma 6, given $\alpha \in [0, 2\pi)$, we can choose a sequence $\{s_d\}$ of positive integers converging to ∞ such that $s_d - m + 1$ is even for each $d \geq 1$ and such that $R_{\theta_1}^{s_d - m + 1} \rightarrow R_\alpha$ and $R_{\theta_j}^{s_d - m + 1} \rightarrow I$ for $2 \leq j \leq k$. It follows that

$$\frac{1}{\binom{s_d}{m-1}} J_m^{s_d}(R_{\theta_1}) \rightarrow \begin{pmatrix} 0 & \dots & 0 & R_\alpha \\ 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

and for any of the other splitting or non-splitting $m \times m$ Jordan block J with $r(J) = 1$, we have

$$\frac{1}{\binom{s_d}{m-1}} J^{s_d} \rightarrow \begin{pmatrix} 0 & \dots & 0 & I \\ 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

For any block J with $r(J) < 1$ or with size smaller than $m \times m$, we have

$$\frac{1}{\binom{s_d}{m-1}} J^{s_d} \rightarrow 0.$$

Arguing as in the proof of (1) \implies (4) in Lemma 6, we see that, if F is the

flip matrix, and S is the matrix that is $\begin{pmatrix} 0 & \cdots & 0 & F \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ on the domain of

$J_m(R_{\theta_1})$, $\begin{pmatrix} 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ on the domains of each of the remaining $m \times m$

blocks J with $r(J) = 1$, and 0 on the domains of the remaining blocks, then $S \in \mathbb{R}\text{-OrbRef}(T)$, but $ST \neq TS$. Hence T is not \mathbb{R} -orbit reflexive.

We need to show that if (2) holds with the condition on θ_1 replaced with condition (3) in Lemma 6, then T must be \mathbb{R} -orbit reflexive. If $m = 1$, then T has the form as in Lemma 6, so we can assume that $m > 1$. Suppose $S \in \mathbb{R}\text{-OrbRef}(T)$ and $0 \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x \end{pmatrix} = X$ is in the domain of $J_m(R_{\theta_1})$. We consider

three cases.

Case 1. $S_1(X) = S(X) = 0$, where S_1 is the restriction of S to the domain of $J_m(R_{\theta_1})$. Suppose Y is orthogonal to the domain of $J_m(R_{\theta_1})$, and using the fact that there is a sequence $\{m_n\}$ of nonnegative integers and a sequence $\{\lambda_n\}$ in \mathbb{R} such that

$$S(X + Y) = \lim_{n \rightarrow \infty} \lambda_n T^{m_n}(X + Y),$$

which means that

$$0 = S(X) = \lim_{n \rightarrow \infty} \lambda_n T^{m_n}(X),$$

and

$$S(Y) = \lim_{n \rightarrow \infty} \lambda_n T^{m_n}(Y).$$

However, the former implies

$$\lim_{n \rightarrow \infty} |\lambda_n| \binom{m_n}{m-1} = 0,$$

which implies $S(Y) = 0$. If $k \geq 2$, then $S_2 = 0$, where S_2 is the restriction of S to the domain of $J_m(R_{\theta_2})$, so the preceding arguments imply that $S_1 = 0$; whence, $S = 0$.

We therefore suppose $k = 1$, and it follows from (3) that $\theta_1/2\pi \in \mathbb{Q}$, i.e., $\theta_1 = 2\pi p/q$ with $1 \leq p < q$ relatively prime integers. We can identify \mathbb{R}^2 with \mathbb{C} , and we can write $x = re^{i\alpha}$ with $r > 0$. Since $S(X) = 0$, we have $S(\frac{1}{r}X) = 0$, so we can assume $x = e^{i\alpha}$. Then $\{\lambda R_{\theta_1}^s x : \lambda \in \mathbb{R}, 1 \leq s \leq q\}$ is the set of all complex numbers whose argument belongs to $\{\alpha + jp2\pi/q : 1 \leq j \leq q\} + \pi\mathbb{Z}$. Choose numbers β and γ with $\alpha < \beta < \gamma < \alpha + \pi/8$ such that

$$[\{\gamma + jp2\pi/q : 1 \leq j \leq q\} + \pi\mathbb{Z}] \cap [\{\beta + jp2\pi/q : 1 \leq j \leq q\} + \pi\mathbb{Z}] = \emptyset.$$

Since the argument of $e^{i\alpha} + te^{i\gamma}$ ranges over (α, γ) as t ranges over $(0, \infty)$, we

can chose $t > 0$ so that the argument of $e^{i\alpha} + te^{i\gamma}$ is β . Now let $W = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ te^{i\gamma} \end{pmatrix}$

in the domain of $J_m(R_{\theta_1})$. Then $S(X + W) = SX + SW = SW$. However, the nonzero coordinates of any vector in the closure of $\mathbb{R}\text{-Orb}(T)(X + W)$ are all complex numbers with arguments in $\{\gamma + jp2\pi/q : 1 \leq j \leq q\} + \pi\mathbb{Z}$ and the nonzero coordinates of any vector in the closure of $\mathbb{R}\text{-Orb}(T)(X + W)$ are all complex numbers with arguments in $\{\beta + jp2\pi/q : 1 \leq j \leq q\} + \pi\mathbb{Z}$. Hence

$S_1 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ y \end{pmatrix} = 0$ for every choice of y . We can apply similar arguments to each of

the other coordinates to get $S_1 = 0$, which implies $S = 0$.

Case 2. $S(X) = S_1(X) = \lambda_0 T^{n_0}(X) \neq 0$. Note that if $\lambda T^s(X) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \neq 0$, then

$$\frac{\|x_{m-1}\|}{\|x_m\|} = s, \text{ and } R_{\theta_1}^{-s} x_m = \lambda x.$$

This means that if $\{m_n\}$ is a sequence of nonnegative integers and $\{\lambda_n\}$ is a sequence in \mathbb{R} , and $T^{m_n}(X) \rightarrow S(X)$, then, eventually $m_n = n_0$ and $\lambda_n \rightarrow \lambda_0$. It follows that $S = \lambda_0 T^{n_0}$ on the orthogonal complement of the domain of S_1 . If $k \geq 2$, we can argue (using S_2) that $S = \lambda_0 T^{n_0}$. If $k = 1$, we can use M_1, M_2, M_3 as in Case 1 to show that $S = \lambda_0 T^{n_0}$.

Case 3. $S(X) = S_1(X) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \neq 0$, but $x_m = 0$. If $\{s_n\}$ is a

sequence of nonnegative integers and $\{\lambda_n\}$ is a sequence in \mathbb{R} and $\lambda_n T^{s_n}(X) \rightarrow S(X)$, we must have $\lambda_n \rightarrow 0$, and thus $s_n \rightarrow \infty$, and $\{|\lambda_n| \binom{s_n}{m-1}\}$ bounded.

Thus $S_1(X) = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. It follows that if J is an $m \times m$ Jordan block of

T with $r(J) = 1$ and whose domain is orthogonal to the domain of S_1 , then the restriction of S to the domain of J is a matrix whose only nonzero entry is in the first row and m^{th} column. The restriction of S to the domain of a block J with $r(J) < 1$ or whose size is smaller than $m \times m$ must be 0. If $k \geq 2$, the S_1 also has an operator matrix whose only nonzero entry is in the first row and m^{th} column. If $k = 1$, then $\theta_1/2\pi$ is rational, and we can argue with M_1, M_2, M_3 as in Case 1 to see that S_1 has a matrix whose only nonzero entry is in the first row and m^{th} column. If the $m \times m$ Jordan blocks of T are $J_m(R_{\theta_1}) \oplus \cdots \oplus J_m(R_{\theta_k}) \oplus J_m(I_a) \oplus J_m(-I_b)$ (I_a is an $a \times a$ identity matrix), then the corresponding decomposition of S is a direct sum of

$\begin{pmatrix} 0 & \cdots & 0 & A_j \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, 1 \leq j \leq k+2$. It is easily seen that $A_1 \oplus \cdots \oplus A_{k+2}$ is

in $\mathbb{R}\text{-OrbRef}(R_{\theta_1} \oplus \cdots \oplus R_{\theta_k} \oplus I_a \oplus -I_b)$. Since $\theta_1, \dots, \theta_k$ satisfy condition (3) in Lemma 6, it follows from Lemma 6 that $R_{\theta_1} \oplus \cdots \oplus R_{\theta_k} \oplus I_a \oplus -I_b$ is \mathbb{R} -orbit reflexive, so there is a sequence $\{s_n\}$ with $s_n \rightarrow \infty$ and a sequence $\{\lambda_n\}$ in \mathbb{R} such that $\lambda_n(R_{\theta_1} \oplus \cdots \oplus R_{\theta_k} \oplus I_a \oplus -I_b)^{s_n-m+1} \rightarrow A_1 \oplus \cdots \oplus A_{k+2}$. Hence

$$\lambda_n T^{s_n} \rightarrow S.$$

Hence T is \mathbb{R} -orbit reflexive. ■

Remark 10 *Using the ideas of the proof of Corollary 8 it is possible to characterize \mathbb{R} -orbit reflexivity for an algebraic operator on a Banach space in terms of its algebraic Jordan form.*

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